Let $\Omega \subset \mathbb{R}^n$ be an open domain and $Y = (0, 1)^n$. Let $f \in L^2(\Omega)$ and $A \in \mathcal{A}_2(\alpha, \beta, \Omega, Y)$, where $A(x, y) = A(y)$. We consider the problem:
Find $u^\varepsilon \in H^1_0(\Omega)$ such that
\begin{equation}
\int_{\Omega} A\left(\frac{x}{\varepsilon}\right) \nabla u^\varepsilon(x) \cdot v(x) \, dx = \int_{\Omega} f(x)v(x) \, dx
\tag{1}
\end{equation}
holds for all $v \in H^1_0(\Omega)$.

**Exercise 1.** (asymptotic expansion I)
We assume that there exist smooth, $Y$-periodic functions $u_i(x, y), i \in \mathbb{N}$ such that
\begin{equation*}
u^\varepsilon(x) = \sum_{i \in \mathbb{N}} \varepsilon^i u_i\left(x, \frac{x}{\varepsilon}\right).
\end{equation*}
Denote with $\text{div}_y, \nabla_y$ the corresponding differential operators with respect to the $y$-variable, as well as $\bar{y} = x/\varepsilon$.

a) Calculate $\nabla u^\varepsilon(x)$ in terms of the $(u_i)'s$, ordered by powers of $\varepsilon$.

b) Plug your result into equation (1) to obtain
\begin{equation*}
\varepsilon^{-2} \left[ \text{div}_y(A(\bar{y})\nabla_y u_0(x, \bar{y})) \right] \\
+ \varepsilon^{-1} \left[ \text{div}_x(A(\bar{y})\nabla_y u_0(x, \bar{y})) + \text{div}_y(A(\bar{y})(\nabla_x u_0(x, \bar{y}) + \nabla_y u_1(x, \bar{y}))) \right] \\
+ \left[ \text{div}_y(A(\bar{y})(\nabla_x u_1(x, \bar{y}) + \nabla_y u_2(x, \bar{y}))) + \text{div}_x(A(\bar{y})(\nabla_x u_0(x, \bar{y}) + \nabla_y u_1(x, \bar{y}))) \right] \\
+ \mathcal{O}(\varepsilon)
\end{equation*}
\begin{equation*} = -f(x).
\end{equation*}
(4 points)

**Exercise 2.** (asymptotic expansion II)
Assume that the equation from Exercise 1b) holds for general $y \in Y$ and equate coefficients to obtain the following differential equations:

- $- \text{div}_y(A(y)\nabla_y u_0(x, y)) = 0$
  with $u_0(x, y)$ is $Y$-periodic. Conclude that $u_0(x, y) = u_0(x)$ is not depending on $y \in Y$.

- $\text{div}_y(A(y)(\nabla_x u_0(x) + \nabla_y u_1(x, y))) = 0$
  with $u_1(x, y)$ is $Y$-periodic. Conclude that one can write
  \begin{equation*}
u_1(x, y) = u_1(x) + \nabla_x u_0(x) \cdot w(y)
\end{equation*}
  with $w : Y \to \mathbb{R}^n$ is $Y$-periodic, where $w_i(y)$ satisfies
  \begin{equation*}
\text{div}_y(A(y)(\nabla_y w_i(y) + e_i)) = 0
\end{equation*}
  for $i = 1, \ldots, n$ (assume that such $w_i$ exist).
\[-f(x) = \text{div}_y(A(y)(\nabla_x u_1(x, y) + \nabla_y u_2(x, y))) + \text{div}_x(A(y)(\nabla_x u_0(x) + \nabla_y u_1(x, y)))\]

Integrate this equation with respect to \(Y\) and show that the first summand on the right hand side vanishes due to a periodicity argument. For the second term, plug in the representation for \(u_1(x, y)\) and conclude that

\[f(x) = -\text{div}_x(A^0_x \nabla_x u_0(x))\]

with

\[A^0_{ij} = \int_Y A(y)(e_j + \nabla_y w_j(y)) \, dy \cdot e_i.\]

(6 points)

**Exercise 3.** (periodic boundary problem)

Let \(Y = (0, 1)^n\), \(f \in L^2(Y)\) and consider the problem:

Find \(u \in \tilde{H}^1(Y)\) such that

\[\int_Y \nabla u(y) \nabla v(y) \, dy = \int_Y f(y)v(y) \, dy\]

for all \(v \in \tilde{H}^1(Y)\).

Here,

\[\tilde{H}^1(Y) = \{ v \in H^1(Y) \mid v \text{ has periodic boundary conditions and zero mean} \}\]

equipped with the usual \(H^1\)-norm. Show that this problem has a unique solution \(u\), which depends continuously on \(f\). Show that if \(u \in C^2(Y)\), it solves the PDE \(-\Delta u = f\) in the strong sense.

(6 points)