Programmieraufgabe 1. (moving least squares)
Let \( \Omega = [0, 1]^2 \) and \( f: \Omega \rightarrow \mathbb{R} \) be given by
\[
f(x) = \begin{cases} 
1 & 1/2 \leq \|x\|_2 \leq 1, \\
0 & \text{else}.
\end{cases}
\]

We want to approximate \( f \) with a smooth function using the moving least squares algorithm.

a) Write a routine that generates \( N \) uniformly distributed random samples \( \{x_i\}_{i=1}^N \subset \Omega \) and stores them in a vector \( x \in (\mathbb{R}^2)^N \).

To allow fast access to the random samples based on their location, we use a binning procedure:

b) Write a routine that takes \( x \in (\mathbb{R}^2)^N \) and \( M \in \mathbb{N} \) as input and produces a set of \( M^2 \) vectors \( y_{ij} \in (\mathbb{R}^2)^n \), \( i, j = 1, \ldots, M \) satisfying (thinking of \( x, y_{ij} \) as sets)
\[
\begin{align*}
- \sum_{i,j=1}^M n_{ij} &= N, \\
- \bigcup_{i,j=1}^M y_{ij} &= x, \\
y_{ij} &\subset \Omega_{ij} = \left[\frac{i-1}{M}, \frac{i}{M}\right] \times \left[\frac{j-1}{M}, \frac{j}{M}\right] \text{ for } i, j = 1, \ldots, M
\end{align*}
\]

As a weight function, we use
\[
\theta(d) = \begin{cases} 
(1 - dM)^4(4dM + 1) & d \leq 1/M, \\
0 & \text{else}.
\end{cases}
\]

Therefore the minimization of
\[
\sum_{i=1}^N \theta(\|z - x_i\|) |f(x_i) - p(x_i)|^2
\]
over a given polynomial space can be performed on the 3x3 Bin-patch surrounding \( z \).

c) Write a routine that takes \( z \in \Omega, M \in \mathbb{N}, \{y_{ij}\}_{i,j=1}^M \) and \( p \in \mathbb{N} \) and returns the MLS-approximation of \( f(z) \) with order \( p \) (we take monomial basis functions \( P_{qr}(z) = z_1^q z_2^r, q + r \leq p \)). This includes the following steps:
\[
- \text{Find } i, j \text{ such that } z \in \Omega_{ij}, \text{ and assemble the points } Z = \bigcup_{k,l \in \{-1,0,1\}} y_{i+k,j+l} = \{z_a\}_{a=1}^m, \\
- \text{assemble the Vandermonde matrix } V \in \mathbb{R}^{m \times (p+1)(p+2)/2}, V_{a,qr} = P_{qr}(z_a) \text{ for } a = 1, \ldots, m, q + r \leq p
\]
assemble the weight matrix $W \in \mathbb{R}^{m \times m}$, $W_{ab} = \delta_{ab} \theta(\|z - z_a\|)$ for $a, b = 1, \ldots, m$

- Solve the linear system
  $$V^TWF = V^TWb$$

  with $F \in \mathbb{R}^m$, $F_a = f(z_a)$ for $a = 1, \ldots, m$. Do this with an iterative solver (e.g. CG-method or Jacobi method) and with the starting point $b^0 = (1/2, 0, \ldots, 0)^T$

- return the MLS approximation $f(z) \approx \sum_{q+r \leq p} b_{qr} P_{qr}(z)$

d) Test your implementation for $N \in \{10, 100, 1000, 10000\}$, $M = \lfloor N^{1/3} \rfloor$, $p = 3$ and plot your solution using an equidistant rectangular sampling grid of size $201 \times 201$.

(20 points)